Lecture 20

Procedure for finding extreme values on closed & bounded sets

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

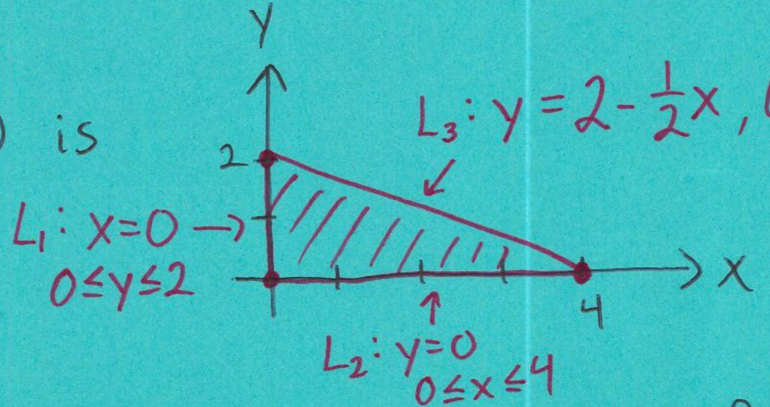
① Find the values of f at critical points of f inside D.

② Find the extreme values of f on the boundary of D.

③ The largest of the values in ① & ② is the absolute maximum value, and the smallest of these values is the absolute minimum value.

Ex: Find the absolute maximum and absolute minimum values of $f(x,y) = x+y-xy$ on the closed and bounded set D which is the closed triangle with vertices $(0,0)$, $(0,2)$, $(4,0)$.

Sol: The region D is



$\nabla f = \langle 1-y, 1-x \rangle$, So the critical point of f is $(1,1)$, which is inside D.

Now, we check on the boundary of D:

$\underline{L_1}$: Plug in $x=0$ to f:

$g_1(y) = f(0,y) = y$. Now we treat g_1 as being defined on $[0,2]$, and do calc I. $g_1'(y)=1$, so it has no critical points. "Check the boundaries" means include the endpoints of L_1 as potential extreme points. The endpoints are $(0,0)$ & $(0,2)$.

$\underline{L_2}$: plug in $y=0$ to f:

$g_2(x) = f(x,0) = x$. Similarly to the above, we only include the endpoints of L_2: $(0,0)$ & $(4,0)$.

$\underline{L_3}$: Plug in $y = 2-\frac{1}{2}x$ to f:

$g_3(x) = f(x, 2-\frac{1}{2}x) = x + (2-\frac{1}{2}x) - x(2-\frac{1}{2}x) = \frac{1}{2}x^2 - \frac{3}{2}x + 2$

$0 \le x \le 4$. $g_3'(x) = x - \frac{3}{2} \Rightarrow x = \frac{3}{2}$ is a critical point of g_3, so we check $\left(\frac{3}{2}, 2-\frac{1}{2}\left(\frac{3}{2}\right)\right) = \left(\frac{3}{2}, \frac{5}{4}\right)$. Also include the boundary points $(4,0)$ & $(0,2)$

Now, we compare them all:

Point	Value	
$(1,1)$	1	
$(0,0)$	0	Abs min
$(0,2)$	2	
$(4,0)$	4	Abs max
$\left(\frac{3}{2},\frac{5}{4}\right)$	$\frac{3}{2}+\frac{5}{4}-\frac{15}{8}=\frac{7}{8}$	

◇

<u>Ex</u>: Find the points on $y^2=9+xz$ which are closest to $(3,0,1)$.

<u>Sol</u>: Let (x,y,z) be a point on the surface. The distance to $(3,0,1)$ from this point is $d=\sqrt{(x-3)^2+(y-0)^2+(z-1)^2}$.

We wish to minimize d. $\quad = \sqrt{(x-3)^2+y^2+(z-1)^2}$

Since $y^2=9+xy$, we can rewrite d as $d=\sqrt{(x-3)^2+9+xz+(z-1)^2}$.

Since d is always positive, minimizing d is equivalent to minimizing $g(x,z)=d^2=(x-3)^2+9+xz+(z-1)^2$.

$$\nabla g=\langle g_x,g_z\rangle=\langle 2x-6+z,\ x+2z-2\rangle$$

The critical points of g are when $\nabla g = \vec{0}$:

$$\begin{cases} 2x-6+z=0 & (1) \\ x+2z-2=0 & (2) \end{cases}$$

(1) $\Rightarrow z=6-2x$ Plug into (2): $x+2(6-2x)-2 = x+12-4x-2=0$

$\Rightarrow -3x+10=0 \Rightarrow x=\frac{10}{3} \Rightarrow z=6-2\left(\frac{10}{3}\right)=\frac{18}{3}-\frac{20}{3}=\frac{-2}{3}$

Now we need to make sure this is a minimum:

$$\text{Mercury} = H_g = \begin{pmatrix} f_{xx} & f_{xz} \\ f_{zx} & f_{zz} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

So, $f_{xx}\left(\frac{10}{3}, \frac{-2}{3}\right)=2$ & $D\left(\frac{10}{3}, \frac{-2}{3}\right)=\det(H_g)\left(\frac{10}{3}, \frac{-2}{3}\right)=4-1=3$.

Since both are positive, it is indeed a minimum.

Now we need to find the y-value(s) of the(se) point(s):

$y^2 = 9+xy = 9+\left(\frac{10}{3}\right)\left(\frac{-2}{3}\right)=\frac{81}{9}-\frac{20}{9}=\frac{61}{9} \Rightarrow y=\pm\frac{\sqrt{61}}{3}$.

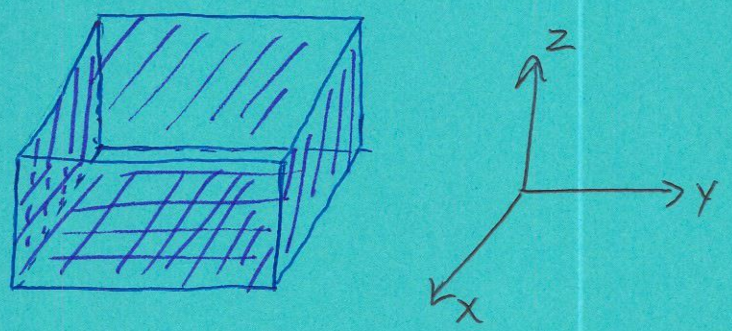
So, the points on $y^2=9+xy$ closest to $(3,0,1)$

are $\left(\frac{10}{3}, \frac{\sqrt{61}}{3}, \frac{-2}{3}\right)$ and $\left(\frac{10}{3}, \frac{-\sqrt{61}}{3}, \frac{-2}{3}\right)$.

$\diamond$

Ex: A cardboard box without a lid is to have a volume of $32000\,cm^3$. Find the dimensions that minimize the amount of cardboard used.

Sol: We have volume is constant, i.e., $V = xyz = 32000$. We aim to minimize surface area. Assume the box is situated as follows:



Then the surface area is $A = xy + 2(xz + yz)$. We need to eliminate one variable, say z: Solve for z from the volume $\Rightarrow z = \dfrac{32000}{xy}$. (Notice here x or $y = 0$ are unacceptable solutions since you can't have a box with a side of length zero!)

$$\Rightarrow A = xy + 2\left(\frac{32000}{y} + \frac{32000}{x}\right) = xy + 64000\left(\frac{1}{x} + \frac{1}{y}\right).$$

$$\nabla A = \left\langle y - \frac{64000}{x^2},\ x - \frac{64000}{y^2} \right\rangle.$$

$\nabla A = \vec{0}$:

$$\begin{cases} y - \dfrac{64000}{x^2} = 0 & (1) \\ x - \dfrac{64000}{y^2} = 0 & (2) \end{cases}$$

$(1) \Rightarrow y = \dfrac{64000}{x^2}$. Plug into (2):

$$x - 64000\frac{1}{y^2} = x - 64000\left(\frac{64000}{x^2}\right)^{-2} = x - \cancel{64000}\left(\frac{x^4}{(64000)^{\cancel{2}}}\right)$$

$= x - \dfrac{x^4}{64000} = x\left(1 - \dfrac{x^3}{64000}\right) = 0$. Since we don't want $x=0$,

the only solution is $1 - \dfrac{x^3}{64000} = 0 \Rightarrow x^3 = 64000 = 40^3 \Rightarrow x = 40$ $(4^3 = 64)$

$\Rightarrow y = \dfrac{64000}{40^2} = \dfrac{40^3}{40^2} = 40$. We make sure this is a minimum:

$$HA = \begin{pmatrix} 3 \cdot \dfrac{64000}{x^3} & 1 \\ 1 & 3 \cdot \dfrac{64000}{y^3} \end{pmatrix}. \quad f_{xx}(40,40) = 3 \cdot \frac{64000}{40^3} = 3 > 0$$

$$D(40,40) = 9 \cdot \frac{(64000)^2}{40^3 40^3} - 1 = 9 \cdot \frac{(40^3)^2}{(40^3)^2} - 1 = 8 > 0.$$

So, $(40,40)$ is a min of A. Use $V = 32000$ to find $z = 20$. So, the dimensions are $40\text{cm} \times 40\text{cm} \times 20\text{cm}$. ◇